The differential coefficient of a sum or difference

is the sum or difference of the differential coefficients of the separate terms.

Thus, if $f(x) = p(x) + q(x) - r(x)$, (where f, p, q and r are functions), then $f'(x) = p'(x) + q'(x) - r'(x)$

Differentiation of common functions is demonstrated in the following worked problems.

> Problem 1. Find the differential coefficients of: (a) $y = 12x^3$ (b) $y = \frac{12}{x^3}$

If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$

(a) Since $y = 12x^3$, $a = 12$ and $n = 3$ thus $\frac{dy}{dx} = (12)(3)x^{3-1} = 36x^2$

(b) $y = \frac{12}{x^3}$ is rewritten in the standard ax^n form as $y = 12x^{-3}$ and in the general rule $a = 12$ and $n = -3$

Thus $\frac{dy}{dx} = (12)(-3)x^{-3-1}$

$= -36x^{-4} = -\frac{36}{x^4}$

PROBLEM 2 Differentiate

$y = 5x^4 + 4x - \frac{1}{2x^2} + \frac{1}{\sqrt{x}} - 3$ with respect to x

SOLUTION

$y = 5x^4 + 4x - \frac{1}{2x^2} + \frac{1}{\sqrt{x}} - 3$ is rewritten as

$y = 5x^4 + 4x - \frac{1}{2}x^{-2} + x^{-1/2} - 3$

When differentiating a sum, each term is differentiated in turn.

$$\text{Thus } \frac{dy}{dx} = (5)(4)x^{4-1} + (4)(1)x^{1-1} - \frac{1}{2}(-2)x^{-2-1} + (1)\left(-\frac{1}{2}\right)x^{(-1/2)-1} - 0$$

$$= 20x^3 + 4 + x^{-3} - \frac{1}{2}x^{-3/2}$$

$$\text{i.e } \frac{dy}{dx} = 20x^3 + 4 - \frac{1}{x^3} - \frac{1}{2\sqrt{x^3}}$$

Differentiation of a product

When $y = uv$, and u and v are both functions of x,

then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

This is known as the **product rule**.

EXAMPLE 1 Find the differential coefficient of

$y = 3x^2 \sin 2x$

SOLUTION

$3x^2 \sin 2x$ is a product of two terms $3x^2$ and $\sin 2x$
Let $u = 3x^2$ and $v = \sin 2x$

Using the product rule:

$$\frac{dy}{dx} = u \quad \frac{dv}{dx} \quad + \quad v \quad \frac{du}{dx}$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow$$

gives: $$\frac{dy}{dx} = (3x^2)(2\cos 2x) + (\sin 2x)(6x)$$

i.e. $$\frac{dy}{dx} = 6x^2 \cos 2x + 6x \sin 2x$$
$$= \mathbf{6x(x\cos 2x + \sin 2x)}$$

EXAMPLE 2 Differentiate: $y = x^3 \cos 3x \ln x$

SOLUTION

Let $u = x^3 \cos 3x$ (i.e. a product) and $v = \ln x$

Then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

where $$\frac{du}{dx} = (x^3)(-3\sin 3x) + (\cos 3x)(3x^2)$$

and $$\frac{dv}{dx} = \frac{1}{x}$$

Hence $$\frac{dy}{dx} = (x^3 \cos 3x)\left(\frac{1}{x}\right) + (\ln x)[-3x^3 \sin 3x + 3x^2 \cos 3x]$$

Differentiation of a quotient

When $y = \frac{u}{v}$, and u and v are both functions of x

then $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

This is known as the **quotient rule**.

EXAMPLE Find the differential coefficient of

$$y = \frac{4\sin 5x}{5x^4}$$

SOLUTION

$\frac{4\sin 5x}{5x^4}$ is a quotient. Let $u = 4\sin 5x$ and $v = 5x^4$

(Note that v is **always** the denominator and u the numerator)

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

where $\frac{du}{dx} = (4)(5)\cos 5x = 20\cos 5x$

and $\frac{dv}{dx} = (5)(4)x^3 = 20x^3$

Hence
$$\frac{dy}{dx} = \frac{(5x^4)(20\cos 5x) - (4\sin 5x)(20x^3)}{(5x^4)^2}$$
$$= \frac{100x^4\cos 5x - 80x^3\sin 5x}{25x^8}$$
$$= \frac{20x^3[5x\cos 5x - 4\sin 5x]}{25x^8}$$

i.e. $\frac{dy}{dx} = \frac{4}{5x^5}(5x\cos 5x - 4\sin 5x)$

CLASS WORK Find the differential coefficient of

1. $\frac{2\cos 3x}{x^3}$ $\left[\frac{-6}{x^4}(x\sin 3x + \cos 3x)\right]$
2. $\frac{2x}{x^2+1}$ $\left[\frac{2(1-x^2)}{(x^2+1)^2}\right]$
3. $\frac{3\sqrt{\theta^3}}{2\sin 2\theta}$ $\left[\frac{3\sqrt{\theta}(3\sin 2\theta - 4\theta\cos 2\theta)}{4\sin^2 2\theta}\right]$

ASSGINMENT Find the differential coefficient of

4. $\frac{\ln 2t}{\sqrt{t}}$ $\left[\frac{1-\frac{1}{2}\ln 2t}{\sqrt{t^3}}\right]$
5. $\frac{2xe^{4x}}{\sin x}$ $\left[\frac{2e^{4x}}{\sin^2 x}\{(1+4x)\sin x - x\cos x\}\right]$